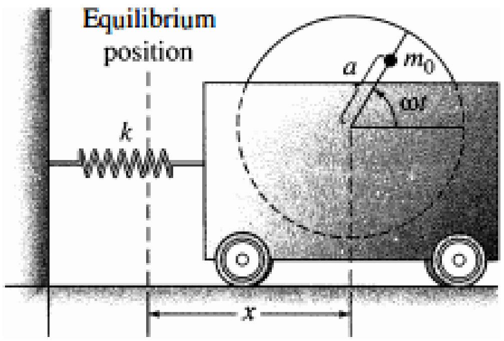
3.6: Forced Oscillations and Resonance

In Section 3.4 we derived the differential equation

$$mx'' + cx' + kx = F(t). \tag{1}$$

We wish now to consider what happens when $F(t) = F_0 \cos \omega t$ or $F(t) = F_0 \sin \omega t$.



An example of when this can occur is when there is a rotating machine component involved in the mass which can provide a simple harmonic force. We arrive at the differential equation

$$mx'' + kx = F_0 \cos \omega t. \tag{2}$$

Undamped Forced Oscillations: To study the undamped oscillations under the influence of the external force $F(t) = F_0 \cos \omega t$, we set $c = 0$ in Equation (1) and begin with the equation

$$mx'' + kx = F_0 \cos \omega t \tag{3}$$

whose complimentary solution is $x_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$, where $\omega_0 = \sqrt{k/m}$ is the **natural frequency** of the mass-spring system. We can also see that the particular solution is of the form $x_p = A \cos \omega t$, where ω is the **circular frequency**. Suppose $\omega \neq \omega_0$. (Why?) Taking derivative of x_p and plugging into Equation (2), we get

$$-m\omega^2 \cos \omega t + kA \cos \omega t = F_0 \cos \omega t$$

so that

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2}. \tag{4}$$

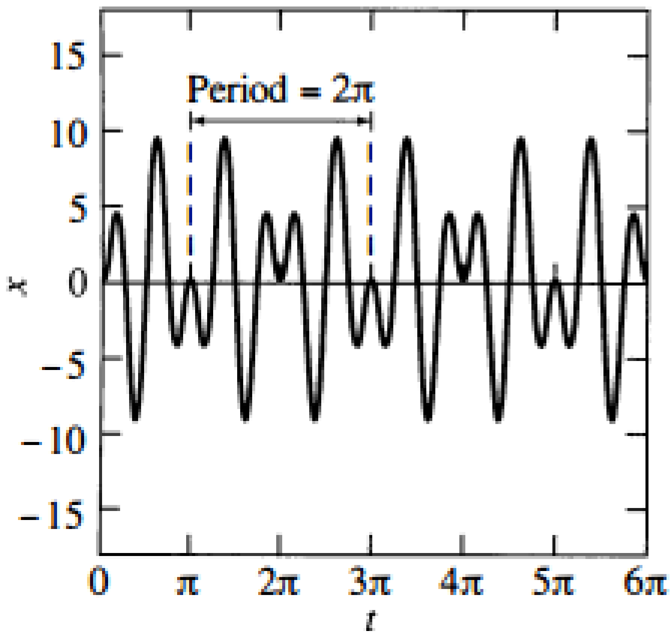
Therefore, the general solution $x = x_c + x_p$ is given by

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t. \tag{5}$$

Just as in Section 3.4, this becomes

$$x(t) = C \cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t. \tag{6}$$

Example 1. (Undamped Forced Oscillations)
Suppose that $m = 1$, $k = 9$, $F_0 = 80$ and $\omega = 5$ in (2). Find $x(t)$ if $x(0) = x'(0) = 0$.



Beats:

If $x(0) = x'(0) = 0$ then the solution to (2) can be arranged as

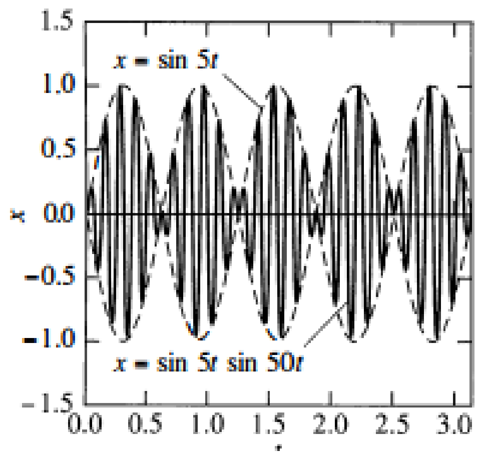
$$\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \tfrac{1}{2}(\omega_0 - \omega)t \sin \tfrac{1}{2}(\omega_0 + \omega)t.$$

We see that if $|\omega - \omega_0|$ is small we get a rapid oscillation plus a slowing varying amplitude.

Example 2. When $m = 0.1$, $F_0 = 50$, $\omega_0 = 55$ and $\omega = 45$ in (2), the solution written as above is given by

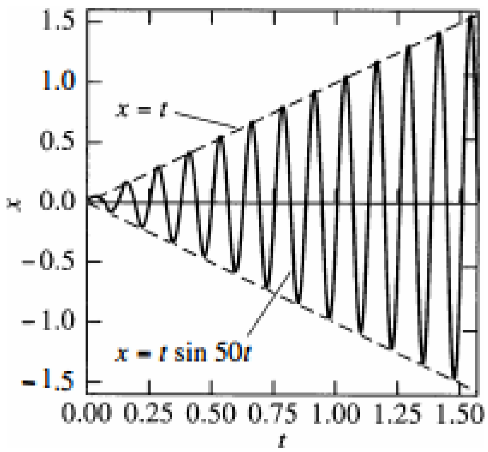
$$x(t) = \sin 5t \sin 50t$$

and the solution curve looks as below.



When $\omega_0 = \omega$ in (2) we see that the complementary and particular solutions would have the same form. In this case we see the phenomenon of resonance.

Example 3. (Resonance) Suppose that in (2) we have that $m = 5$ kg and $k = 500$ N/m. Then the natural frequency is $\omega_0 = 10$ rad/s. If the flywheel revolves at the same rate, then the solution curve looks as below.



More Complex Examples:

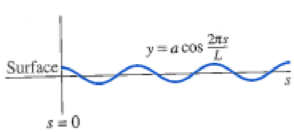


FIGURE 3.6.6. The washboard road surface of Example 5.

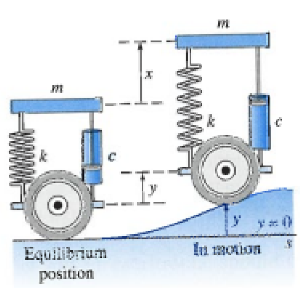


FIGURE 3.6.7. The "unicycle model" of a car.

Damped Forced Oscillations: Consider now the full generality of Equation (2):

$$mx'' + cx' + kx = F_0 \cos \omega t.$$

In this case, we can apply the same trig laws as in Section 3.4 to get

$$x_p = C \cos(\omega t - \alpha).$$

Example 4. Find the transient motion (x_c) and the steady periodic oscillations (x_p) of a damped mass-and-spring system with $m = 1$, $c = 2$, and $k = 26$ under the influence of an external force $F(t) = 82 \cos 4t$ with $x(0) = 6$ and $x'(0) = 0$. Also investigate the possibility of practical resonance for this system; i.e. what values of ω maximize the forced amplitude?

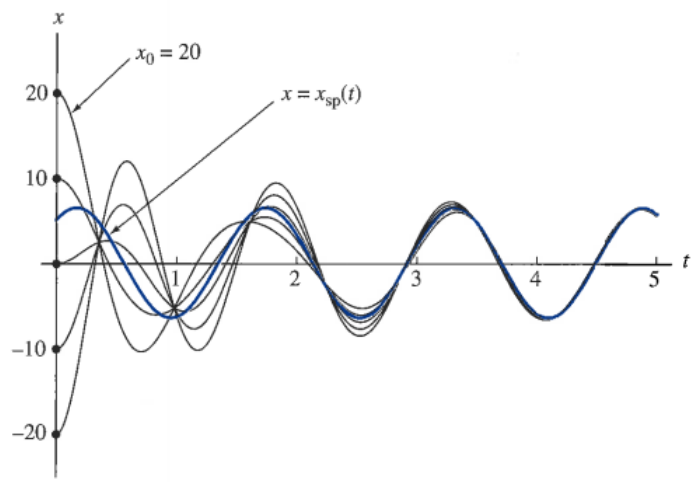


FIGURE 3.6.8. Solutions of the initial value problem in (24) with $x_0 = -20, -10, 0, 10$, and 20.

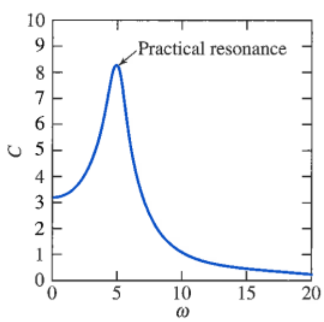


FIGURE 3.6.9. Plot of amplitude C versus external frequency ω.

To investigate the possibility of practical resonance in the given system, we substitute the values $m = 1$, $c = 2$, and $k = 26$ in (21) and find that the forced amplitude at frequency ω is

$$C(\omega) = \frac{82}{\sqrt{676 - 48\omega^2 + \omega^4}}.$$

The graph of $C(\omega)$ is shown in Fig. 3.6.9. The maximum amplitude occurs when

$$C'(\omega) = \frac{-41(4\omega^3 - 96\omega)}{(676 - 48\omega^2 + \omega^4)^{3/2}} = \frac{-164\omega(\omega^2 - 24)}{(676 - 48\omega^2 + \omega^4)^{3/2}} = 0.$$

Thus practical resonance occurs when the external frequency is $\omega = \sqrt{24}$ (a bit less than the mass-and-spring's undamped critical frequency of $\omega_0 = \sqrt{k/m} = \sqrt{26}$). ■

Homework. 1-6 (all)